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Curing the Curse of Non-Recursiveness in Structural Causal Models

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CausAM

Abstract

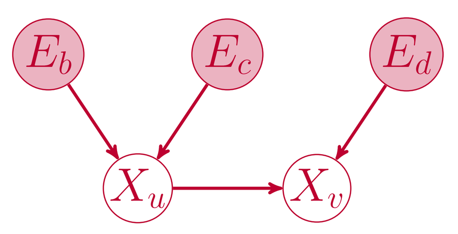
In this work (Bongers et al., 2016) we give a general treatment of **structural causal models (SCMs)** in the **cyclic** setting. We show that if one is only interested in a particular subset of endogenous variables, one can under certain conditions arrive at a more parsimonious representation of the SCM by means of **marginalizing** out endogenous and **reducing** exogenous variables.

Structural Causal Models

$$\textbf{SCM} = \begin{cases} \textbf{Structural equations}: & \boldsymbol{x} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{e}), \\ \textbf{Exogenous distribution}: & \mathbb{P}_{\boldsymbol{\mathcal{E}}}. \end{cases}$$

Example:

$$f_u(\boldsymbol{x},\boldsymbol{e}) = e_b + e_c$$
$$f_v(\boldsymbol{x},\boldsymbol{e}) = x_u + x_j + e_d$$
$$p(e_b, e_c, e_d) = \mathcal{N}(\boldsymbol{0}, \boldsymbol{I})$$



An SCM is **uniquely solvable** if the structural equations have a unique solution. Every acyclic SCM is uniquely solvable. Extending the above model to the SCM $\mathcal{M}$:

$$f_h(\boldsymbol{x},\boldsymbol{e}) = x_u + x_j + e_a$$
$$f_i(\boldsymbol{x},\boldsymbol{e}) = x_j + e_b$$
$$f_j(\boldsymbol{x},\boldsymbol{e}) = \alpha x_i$$

leads to a non-uniquely solvable SCM for $\alpha = 1$.

Unique solvability implies that the SCM has a **solution** in terms of random variables $(\boldsymbol{X}, \boldsymbol{E})$:

(1) $\mathbb{P}^{\boldsymbol{E}} = \mathbb{P}_{\boldsymbol{\mathcal{E}}}$

(2) $\boldsymbol{X} = \boldsymbol{f}(\boldsymbol{X}, \boldsymbol{E})$ a.s..

that has a unique distribution.

Perfect interventions modelled à la Pearl. The perfect intervention $\mathrm{do}(j, 5)$ on $\mathcal{M}$ changes the causal mechanism of f_j to

$$f_j(\boldsymbol{x},\boldsymbol{e}) = 5$$

which yields a uniquely solvable SCM $\mathcal{M}_{\mathrm{do}(j,5)}$.

Cyclic SCMs are challenging, since unique solvability is not guaranteed (not even preserved under perfect interventions).

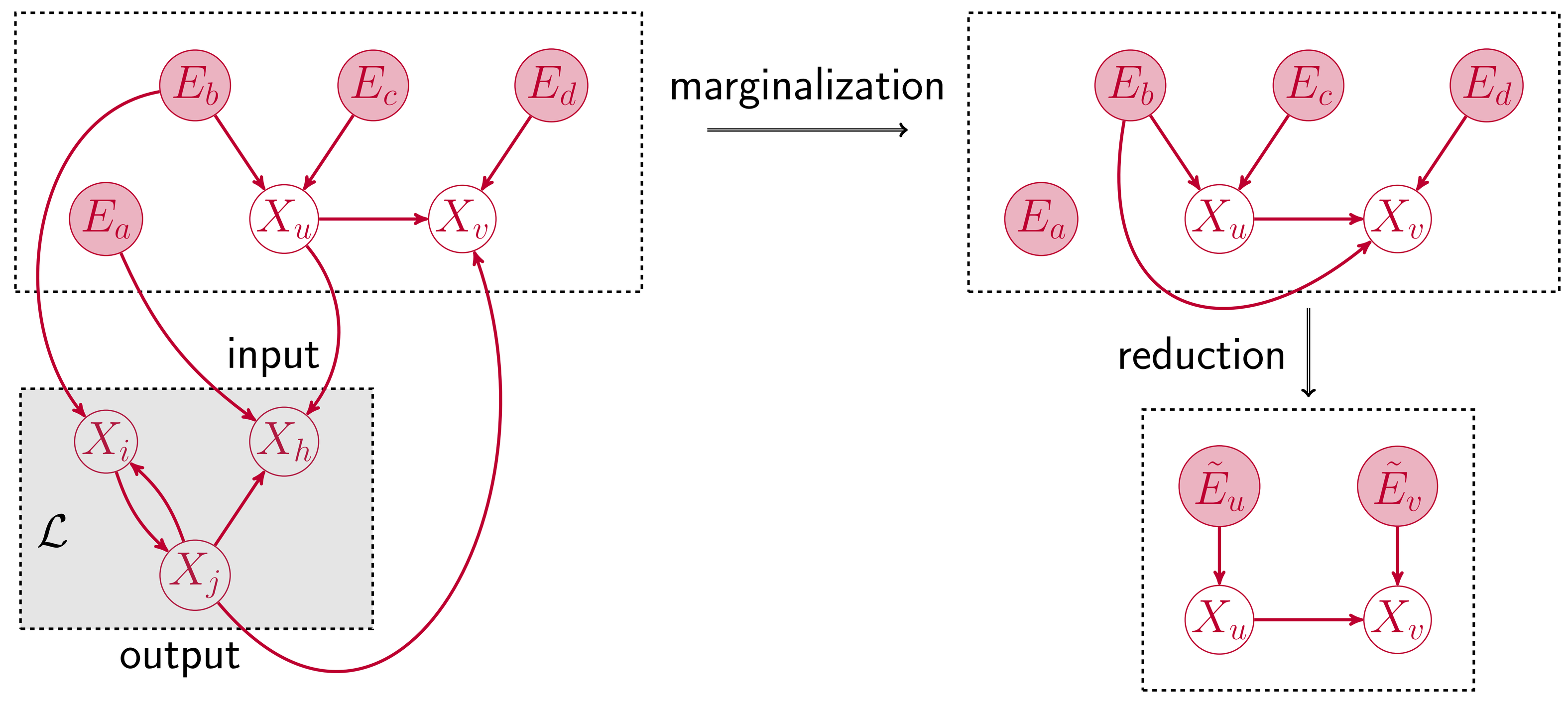
Marginalization

Define the **marginalization** $\mathcal{M}_{\mathrm{marg}(\mathcal{L})}$ over the subset of endogenous variables $\mathcal{L}$ by

$$\boldsymbol{f}_{\mathrm{marg}(\mathcal{L})}(\boldsymbol{x},\boldsymbol{e}) := \boldsymbol{f}\big(\boldsymbol{g}_{\mathcal{L}}(\boldsymbol{x},\boldsymbol{e}), \boldsymbol{x}, \boldsymbol{e}\big)$$

where $\boldsymbol{g}_{\mathcal{L}}$ is the induced mapping obtained by solving the structural equations of $\boldsymbol{f}_{\mathcal{L}}$ for the variables $\mathcal{L}$. For $\alpha = \frac{1}{2}$ this gives

$$\begin{aligned} x_h &= x_u + e_a + e_b \\ x_i &= 2e_b \\ x_j &= e_b \end{aligned} \quad \Longrightarrow \quad \begin{aligned} g_h(\boldsymbol{x},\boldsymbol{e}) &= x_u + e_a + e_b \\ g_i(\boldsymbol{x},\boldsymbol{e}) &= 2e_b \\ g_j(\boldsymbol{x},\boldsymbol{e}) &= e_b \end{aligned}$$



Theorem 1 *Let $\mathcal{M}$ be an SCM that is uniquely solvable w.r.t. $\mathcal{L}$. Then $\mathcal{M}$ and its marginalization $\mathcal{M}_{\mathrm{marg}(\mathcal{L})}$ are interventionally equivalent w.r.t. the complement of $\mathcal{L}$.*

Exogenous Reparametrization

Define the **exogenous reparametrization** $\mathcal{M}_{\mathrm{rep}(\boldsymbol{\phi})}$ with respect to a reparameterization $\boldsymbol{\phi}$ by $\boldsymbol{f}_{\mathrm{rep}(\boldsymbol{\phi})}$ such that

$$\boldsymbol{f}_{\mathrm{red}(\boldsymbol{\phi})}(\boldsymbol{x}, \boldsymbol{\phi}(\boldsymbol{e})) = \boldsymbol{f}(\boldsymbol{x},\boldsymbol{e}).$$

In the example, this gives:

$$\begin{aligned} \phi_u(\boldsymbol{e}) &= e_b + e_c \\ \phi_v(\boldsymbol{e}) &= e_b + e_d \end{aligned} \quad \Longrightarrow \quad \begin{aligned} f_{\mathrm{rep}(\boldsymbol{\phi}),u}(\boldsymbol{x},\tilde{\boldsymbol{e}}) &= \tilde{e}_u \\ f_{\mathrm{rep}(\boldsymbol{\phi}),v}(\boldsymbol{x},\tilde{\boldsymbol{e}}) &= x_u + \tilde{e}_v \end{aligned}$$

Theorem 2 *An exogenously reparametrized SCM is interventionally equivalent to the original SCM.*

Corollary 1 *Every real-valued SCM $\mathcal{M}$ can be exogenously reparametrized to a real-valued SCM $\mathcal{M}_{\mathrm{rep}(\boldsymbol{\phi})}$ with only a single one-dimensional real-valued exogenous variable.*

Reduction

A **reduction** $\mathcal{M}_{\mathrm{red}}$ is an interventionally equivalent SCM with a smaller space of exogenous variables.

By Corollary 1, **reductions** always exist, however $\boldsymbol{f}_{\mathrm{red}}$ is typically very wild (hard to estimate from data!).

We introduce a class of **nice reductions** that generalize smooth reductions of linear models and Markovian models.

Theorem 3 *Nice reductions do not exist in general.*

For example, no nice reduction exists if we change f_v to:

$$f_v(\boldsymbol{x},\boldsymbol{e}) = x_u x_j + e_d \quad \Longrightarrow \quad \begin{aligned} f_{\mathrm{marg}(\mathcal{L}),u}(\boldsymbol{x},\boldsymbol{e}) &= e_b + e_c \\ f_{\mathrm{marg}(\mathcal{L}),v}(\boldsymbol{x},\boldsymbol{e}) &= x_u e_b + e_d \end{aligned}$$

Discussion

What parameterisations of SCMs should we use, given that nice reductions do not exist?

References

Bongers, S., Peters, J., Schölkopf, B., and Mooij, J. M. (2016). Structural causal models: Cycles, marginalizations, exogenous reparametrizations and reductions. *arXiv.org preprint*, arXiv:1611.06221 [stat.ME].

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